Example: Find the co-ordinate vector of $\mathbf{W}=(1,1)$ relative to the basis $\{\mathbf{u}_1=(1,-1),\ \mathbf{u}_2=(1,1)\}$ for R^2.

Solution: To find $(\mathbf{W})_S$, we must first express $\mathbf{W}$ as a linear combination of vectors in basis ie:, we must find values c_1 & c_2 such that

$$\mathbf{W} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

ie:, $$(1,1) = c_1(1,-1) + c_2(1,1)$$

$$\Rightarrow \quad (1,1) = (c_1+c_2,\ -c_1+c_2)$$

Equating the components on both sides

$$c_1 + c_2 = 1 \quad \text{——(i)}$$
$$-c_1 + c_2 = 1 \quad \text{——(ii)}$$

Solving (i) & (ii), we get

$$c_1 = 0,\quad c_2 = 1$$

Thus, the co-ordinate vector of $\mathbf{W}$ relative to basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $(0,1)$.

SEC (4.7) ROW SPACE, COLUMN SPACE AND NULL SPACE

In this Section, we will study some important vector spaces that are associated with matrices.

Definition: for an $m \times n$ matrix
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors $r_1 = [a_{11} \; a_{12} \cdots a_{1n}]$, $r_2 = [a_{21} \; a_{22} \cdots a_{2n}], \ldots, r_m = [a_{m1} \; a_{m2} \cdots a_{mn}]$ in R^n that are formed from the rows of A are called Row Vectors of A.

and the vectors $c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, $\ldots$, $c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

in R^m formed from the column of A are called Column Vectors of A.

Example ① Row and Column Vectors of a 2×3 matrix

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}_{2\times 3}$$

The row vectors of A are $r_1 = [2 \;\; 1 \;\; 0]$ and $r_2 = [3 \;\; -1 \;\; 4] \in R^3$

and the column vectors of A are $c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \in R^2$

Definition: If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the Row space of A, and the subspace of R^m spanned by the column vectors of A is called the Column Space of A.

The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of R^n, is called the Null Space of A.

In this Section and the next, we will be concerned with two general questions—

Q① What relationships exist among the solutions of a linear system $Ax = b$ and the row space, column space and null space of the coefficient matrix A?

Q② What relationships exist among row space, column space & null space of a matrix?

Starting with the first question, suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It follows from Sec 1.3 that if $c_1, c_2, \ldots, c_n$ denote the column vectors of A, then the product $A\mathbf{x}$ can be expressed as a linear combination of these vectors with coefficients from $\mathbf{x}$; that is,

$$A\mathbf{x} = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n \qquad \text{———} \; (1)$$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of m equations in n unknowns can be written as

$$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = \mathbf{b} \qquad \text{———} \; (2)$$

~~from~~ from which we conclude that $A\mathbf{x} = \mathbf{b}$ is Consistent iff $\mathbf{b}$ is expressible as a linear combination of the column vectors of A. This yields the following theorem –

THEOREM 1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is Consistent if and only if $\mathbf{b}$ is in the column space of A.

Example 2 **A Vector 'b' in the Column Space of 'A'**

Let $A\mathbf{x} = \mathbf{b}$ be a linear system
$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Soln. The Augmented matrix, $[A|b] = \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ -1 & 3 & 2 & 1 \\ 2 & 1 & -2 & -3 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ 0 & 5 & -1 & -8 \\ 0 & -3 & 4 & 15 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & -3 & 4 & 15 \end{array}\right] \quad R_2 \to \tfrac{1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -9 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 17/5 & 51/5 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

The corresponding equations are

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -9 \\ x_2 - \tfrac{1}{5}x_3 &= -\tfrac{8}{5} \\ \tfrac{17}{5}x_3 &= \tfrac{51}{5} \end{aligned}$$

Solving these equations, $x_1 = 2$, $x_2 = -1$, $x_3 = 3$

It follows from this and formula ② that

$$x_1c_1 + x_2c_2 + x_3c_3 = 2\begin{bmatrix}-1\\1\\2\end{bmatrix} + (-1)\begin{bmatrix}3\\2\\1\end{bmatrix} + 3\begin{bmatrix}2\\-3\\-2\end{bmatrix}$$

$$= \begin{bmatrix}-2\\2\\4\end{bmatrix} + \begin{bmatrix}-3\\-2\\-1\end{bmatrix} + \begin{bmatrix}6\\-9\\-6\end{bmatrix}$$

$$= \begin{bmatrix}-2-3+6\\2-2-9\\4-1-6\end{bmatrix}$$

$$= \begin{bmatrix}1\\-9\\-3\end{bmatrix} = \mathbf{b}$$

NOTE: Recall from a Theorem of Sec 3.4 that the general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of this system to the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. Keeping in mind that the Null Space of A is the same as the soln. space of $A\mathbf{x} = \mathbf{0}$, we can rephrase that theorem in the following vector form —

THEOREM ② If $\mathbf{x}_0$ is any soln. of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every soln. of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \qquad \text{—— ③}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a soln. of $A\mathbf{x} = \mathbf{b}$.

NOTE. Eqn. ③ gives a formula for the general soln. of $A\mathbf{x} = \mathbf{b}$. The vector $\mathbf{x}_0$ in that formula is called a Particular Soln. of $A\mathbf{x} = \mathbf{b}$ and the remaining part of the formula is called the General Soln. of $A\mathbf{x} = \mathbf{0}$. In words,

The General Soln. of a consistent linear system can be expressed as the sum of a Particular Soln. of that system and General Soln. of the corresponding homogeneous system.

BASES FOR ROW SPACES, COLUMN SPACES AND NULL SPACES

We first developed elementary row operations for the purpose of solving linear systems and we know from that work that performing an elementary row operation on an augmented matrix does not change the soln. set of the corresponding linear system. It follows that applying an elementary row operation to a matrix A does not change the soln. set of the corresponding linear system $Ax=0$, or, stated another way, it does not change the null space of A. Thus, we have the following theorem —

THEOREM 3 Elementary row operations do not change the null space of a matrix.

The following theorem is a companion to Theorem 3 :—

THEOREM 4 Elementary row operations do not change the Row Space of a matrix.

NOTE: Theorems 3 and 4 might tempt you into incorrectly beleiving that elementary row operations do not change the column space of a matrix.

To see why this is not true, ~~compt~~

compare the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

The matrix B can be obtained from A by adding -2 times the first row to the second row. However, this operation has changed the column space of A, since that column space consists of all scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

whereas the column space of B consists of all scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the two are different spaces.

The following theorem makes it possible to find bases for the row and column spaces of a matrix in row echelon form by inspection.

THEOREM (5) If a matrix R is in row echelon form, then the row vectors with the leading 1's (the non-zero row vectors) form a basis for the row space of R and the column vectors with leading 1's of the row vectors form a basis for column space of R.

Example (4) **Bases for Row and Column Spaces**

The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is in row echelon form.

From Theorem (5), the vectors

$$r_1 = [1 \quad -2 \quad 5 \quad 0 \quad 3]$$
$$r_2 = [0 \quad 1 \quad 3 \quad 0 \quad 0]$$
$$r_3 = [0 \quad 0 \quad 0 \quad 1 \quad 0]$$

form a basis for the row space of R.

and the vectors $c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

form a basis for the column space of R.

Example (5) **Basis for a Row Space by Row Reduction**

Find a basis for the row space of the matrix $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & -3 & -4 & 2 & -5 & -4 \end{bmatrix}$

Soln. Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row echelon form of A. Reducing A to row echelon form as

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

By Theorem (5), the non-zero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A.

These basis vectors are

$$r_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$
$$r_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$
$$r_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

THEOREM ⑥ If A and B are row equivalent matrices, then —

(i) A given set of column vectors of A is linearly independent iff the corresponding column vectors of B are linearly independent.

(ii) A given set of column vectors of A forms a basis for the column space of A iff the corresponding column vectors of B form a basis for the column space of B.

Example ⑥ Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$

Soln. We observe in Example ⑤ that the matrix

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is a row echelon form of A.

Keeping in mind that A and R can have different column spaces, we cannot find a basis for the column space of A directly from the column vectors of R. However, it follows from Theorem 6 (ii), that if we can find a set of column vectors of R that forms a basis for the column space of R, then the corresponding column vectors of A will form a basis for the column space of A.

Since the first, third and fifth columns of R contain leading 1's of the row vectors, the vectors $c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

form a basis for the column space of R.

Thus, the corresponding column vectors of A, which are

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

DIMENSION OF ROW SPACE : The dimension of row space is the no. of basis vectors for the row space of A, thus the dimension of row space is the no. of non-zero rows in echelon form.

Example: find the dimension of the row space of $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

Solu: Reducing the matrix A to row echelon form as —

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 4 & -5 \\ 0 & 2 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & -5/4 \\ 0 & 2 & -3 \end{bmatrix} \quad R_2 \to \frac{1}{4} R_2$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & -5/4 \\ 0 & 0 & -1/2 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & -5/4 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to (-2) R_3$$

$\therefore$ dimension of row space of A = no. of non-zero rows in row echelon form

= 3.

SEC 4.8 RANK, NULLITY AND FUNDAMENTAL MATRIX SPACES.

In the last Section, we investigated relationships between a system of linear equations and the row space, column space and null space of its coefficient matrix. In this Section, we will be concerned with the dimensions of those spaces.

Row and Column Spaces Have Equal Dimensions:

In examples 5 and 6 of Section 4.7, we found that the row and column spaces of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

both have three basis vectors and hence both spaces are three dimensional. The fact that these spaces have the same dimension is not accidental, but rather a consequence of the following theorem —

THEOREM 1 The Row Space and Column Space of a matrix A have same dimension.

RANK AND NULLITY: The dimensions of the row space, column space and null space of a matrix are such important numbers that there is some notation and terminology associated with them.

Definition: The common dimension of the row space and column space of a matrix A is called the Rank of A and is denoted by rank(A); the dimension of the null space of A is called the Nullity of A and is denoted by nullity(A).

NOTE: The rank of A can be interpreted as the no. of leading 1's in any row echelon form of the matrix A.

Example 1 Rank and Nullity of a 4X6 Matrix

Find the rank and nullity of the matrix $A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$

Soln.

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \quad R_1 \to R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 2 & -4 & -24 & -32 & 10 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 3 & -6 & -36 & -48 & 15 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 2 & -4 & -24 & -32 & 10 \\ 0 & 3 & -6 & -36 & -48 & 15 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

~~Which is reduced row echelon form of A~~

$$\sim \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + 3R_2 \quad \text{—— (1)}$$

which is reduced row echelon form of A.

Since this matrix has two leading 1's, its row and column spaces are two-dimensional and so rank$(A) = 2$.

To find the nullity of A, we must find the dimension of the solution space of the linear system $A\mathbf{x} = 0$

This system can be solved by reducing its Augmented matrix to reduced row echelon form

$$[A|b] = \left[\begin{array}{cccccc|c} -1 & 2 & 0 & 4 & 5 & -3 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 0 & -4 & -28 & -37 & 13 & 0 \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

Solving these equations for leading variables (corresp. to leading 1's) yields

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6 \quad \text{—— (2)}$$
$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 \quad \text{—— (3)}$$

Let $x_3 = r$, $x_4 = s$, $x_5 = t$ & $x_6 = u$

then from (3), $x_2 = 2r + 12s + 16t - 5u$

& from (2), $x_1 = 4r + 28s + 37t - 13u$

Thus the general solu. in column vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4r + 28s + 37t - 13u \\ 2r + 12s + 16t - 5u \\ r + 0s + 0t + 0u \\ 0r + s + 0t + 0u \\ 0r + 0s + t + 0u \\ 0r + 0r + 0t + u \end{bmatrix}$$

$$= r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{— ④}$$

Because the four vectors on R.H.S. of ④ form a basis for solu. space, nullity$(A) = 4$.

Example ② Maximum Value of Rank

What is the maximum possible rank of an $m \times n$ matrix A that is not square?

Solu. Since the row vectors of A lie in R^n and the column vectors in R^m, the row space of A is at most n-dimensional and the column space is at most m-dimensional. Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n.

We denote this by writing $\quad \text{rank}(A) \le \min(m, n)$.

The following theorem establishes an important relationship between the Rank and Nullity of a matrix —

THEOREM ② Dimension Theorem for Matrices

If A is a matrix with n columns, then $\quad \text{Rank}(A) + \text{Nullity}(A) = n$.

Example ③ The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so $\quad \text{rank}(A) + \text{nullity}(A) = 6$

This is consistent with Example ①, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4.$$

The following theorem, which summarizes the results already obtained, interprets the rank and nullity in the context of a homogeneous linear system —

THEOREM (3) If A is an $m \times n$ matrix, then

(i) rank(A) = the no. of leading variables in the general soln. of $A\mathbf{x} = \mathbf{0}$.

(ii) nullity(A) = the no. of parameters in the general soln. of $A\mathbf{x} = \mathbf{0}$.

Example (4) No. of Parameters in a General Solution

Find the no. of parameters in the general soln. of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.

Soln: We know that rank(A) + nullity$(A) = n$, where n is no. of columns in matrix A.

i.e., $3 + \text{nullity}(A) = 7$

$$\Rightarrow \text{nullity}(A) = 4.$$

Thus, there are 4 parameters in the general soln. of $A\mathbf{x} = 0$.

Equivalence Theorem: In a Theorem of Sec (2.3), we listed seven results that are equivalent to the invertibility of a square matrix A. We are now in a position to add eight more results to that list to produce a single theorem that summarizes most of the topics we have covered thus far —

THEOREM (4) Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent —

(i) A is invertible.

(ii) $A\mathbf{x} = \mathbf{0}$ has only trivial solution.

(iii) The reduced row echelon form of A is I_n.

(iv) A is expressible as a product of elementary matrices.

(v) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(vi) $A\mathbf{x} = \mathbf{b}$ has exactly one soln. for every $n \times 1$ matrix $\mathbf{b}$.

(vii) $\det(A) \neq 0$.

(viii) The column vectors of A are linearly independent.

(ix) The row vectors of A are linearly independent.

(x) The column vectors of A span R^n.

(xi) The row vectors of A span R^n.

(xii) The column vectors of A form a basis for R^n.

(xiii) The row vectors of A form a basis for R^n.

(xiv) A has rank n.

(xv) A has nullity 0.

THEOREM (5) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general soln. of the system contains $n-r$ parameters.

The Fundamental Spaces of a Matrix

There are six important vector spaces associated with a matrix A and its transpose A^T :-

row space of A	row space of A^T
column space of A	column space of A^T
null space of A	null space of A^T

However, transposing a matrix converts row vectors into column vectors and conversely, so except for a difference in notation, the row space of A^T is the same as the column space of A, and the column space of A^T is the same as the row space of A. Thus, of the six spaces listed above, only the following four are distinct :-

row space of A	column space of A
null space of A	null space of A^T

If A is an $m \times n$ matrix, then the row space and null space of A are the subspaces of R^n, and the column space of A and the null space of A^T are the subspaces of R^m.

These are called the Fundamental Spaces of a matrix A.

We will conclude this Section by discussing how these four subspaces are related.

Let us focus for a moment on the matrix A^T. Since the row space and column space of a matrix have the same dimension and since transposing a matrix converts its columns to rows and its rows to columns, the following result should not be surprising

THEOREM (7) If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

NOTE: This result has some important implications.

For example, if A is an $m \times n$ matrix, then A^T will be $n \times m$ matrix so applying formula of Theorem (2) to the matrix A^T,

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

$$\Rightarrow \text{rank}(A) + \text{nullity}(A^T) = m \text{ , using Theorem (7)} \qquad \text{— (1)}$$

This alternative ~~formula~~ form of formula in Theorem (2) makes it possible to express the dimensions of all four fundamental spaces in terms of the size and rank of A. Specifically, if $\text{rank}(A) = r$, then

$$\left.\begin{array}{ll} \dim[\text{row}(A)] = r & \dim[\text{col}(A)] = r \\ \dim[\text{null}(A)] = n-r & \dim[\text{null}(A^T)] = m-r \end{array}\right\} \qquad \text{— (2)}$$

The four formulas in (2) provide an algebraic relationship between the size of a matrix and the dimensions of its fundamental spaces.

SEC 4.9 MATRIX TRANSFORMATIONS FROM R^n to R^m

In this Section, we will study functions of the form $\mathbf{w} = F(\mathbf{x})$, where the independent variable $\mathbf{x}$ is a vector in R^n and the dependent variable $\mathbf{w}$ is a vector in R^m. We will concentrate on a special class of such functions called "Matrix Transformations." Such transformations are fundamental in the study of linear algebra and have important applications in physics, engineering etc.

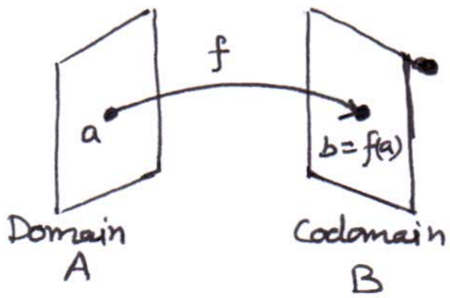
Functions and Transformations: Recall that a function is a rule that associates with each element of a set A one and only one element in a set B.

If f associates the element 'b' with the element 'a', then we write

$$b = f(a)$$

and we say that 'b' is the Image of 'a' under f or that $f(a)$ is the value of f at a. The set A is called the Domain of f and the set B is called the Co-domain of f (See Fig.)

The subset of the codomain that consists of all images of points in the domain is called the Range of f.



For many common functions, the domain and co-domain are sets of real numbers, but in this text we will be concerned with functions for which the domain and codomain are vector spaces.

Definition: If V and W are vector spaces and if f is a function with domain V and co-domain W, then we say that f is a Transformation from V to W or that f maps V to W, which we denote by writing $f: V \to W$.

In the special case where $V = W$, the transformation is also called an Operator on V.

In this Section, we will be concerned exclusively with transformations from R^n to R^m, transformations from general vector spaces will be considered in a later Section. To illustrate one way in which such transformations can arise, suppose that $f_1, f_2, \cdots, f_m$ are real-valued functions of n variables, say

$$\left.\begin{aligned} w_1 &= f_1(x_1, x_2, \cdots, x_n) \\ w_2 &= f_2(x_1, x_2, \cdots, x_n) \\ &\ \vdots \\ w_m &= f_m(x_1, x_2, \cdots, x_n) \end{aligned}\right\} \quad \text{——} \quad (1)$$

These m equations assign a unique point $(w_1, w_2, \cdots, w_m)$ in R^m to each point $(x_1, x_2, \cdots, x_n)$ in R^n and thus define a transformation from R^n to R^m.
If we denote this transformation by T, then $T: R^n \to R^m$

and $\quad T(x_1, x_2, \cdots, x_n) = (w_1, w_2, \cdots, w_m)$

MATRIX TRANSFORMATIONS

In the special case where the equations in ① are linear, they can be expressed in the form

$$\left.\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}\right\} \text{——} \ ②$$

which can be written in matrix notation as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{——} \ ③$$

or more briefly as $\quad \mathbf{w} = A\mathbf{x} \quad$ —— ④

Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector $\mathbf{x}$ in R^n into the column vector $\mathbf{w}$ in R^m by multiplying $\mathbf{x}$ on the left by A. We call this a Matrix Transformation (or Matrix Operator if $m=n$) and we denote it by $T_A: R^n \to R^m$.

With this notation equ. ④ can be expressed as $\quad \mathbf{w} = T_A(\mathbf{x}) \quad$ —— ⑤

The matrix transformation T_A is called Multiplication by A, and the matrix A is called the Standard Matrix for the transformation.

We will also find it convenient to express ⑤ in the schematic form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w} \quad \text{——} \ ⑥$$

which is read as 'T_A maps $\mathbf{x}$ into $\mathbf{w}$'.

Example ① A Matrix Transformation from R^4 to R^3

The matrix transformation $T: R^4 \to R^3$ defined by the equations

$$\left.\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned}\right\} \quad \text{--- ⑦}$$

can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{--- ⑧}$$

so the standard matrix for T is

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

The image of a point (x_1, x_2, x_3, x_4) can be computed directly from the defining equations ⑦ or from ⑧ by matrix multiplication.

For example, if $(x_1, x_2, x_3, x_4) = (1, -3, 0, 2)$

then substituting in ⑦ yields $w_1 = 1$, $w_2 = 3$, $w_3 = 8$

or alternatively from ⑧,

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9+0-10 \\ 4-3+0+2 \\ 5+3+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

Example ② If $T(x_1, x_2) = (x_1 - x_2,\ 2x_1,\ 3x_2 + x_1)$, then find

i) the domain of T (ii) the codomain of T (iii) the image $(1, -2)$

Soln. Here it is clear that $T: R^2 \to R^3$ since $T(x_1, x_2) = (x_1 - x_2,\ 2x_1,\ 3x_2 + x_1)$

so i) the domain of T is R^2

ii) the codomain of T is R^3

iii) The image of $(1, -2)$ is given as

$$T(1, -2) = \big(1 - (-2),\ 2(1),\ 3(-2) + 1\big)$$

$$= (3, 2, -5)$$

Some Notational Matters

Sometimes we want to denote a matrix transformation without giving a name to the matrix itself. In such cases, we will denote the standard matrix for $T: R^n \to R^m$ by the symbol $[T]$. Thus, the equ.

$$T(\mathbf{x}) = [T]\,\mathbf{x} \qquad \text{———} \quad (9)$$

is simply the statement that T is a matrix transformation with standard matrix $[T]$ and the image of $\mathbf{x}$ under this transformation is the product of the matrix $[T]$ and the column vector $\mathbf{x}$.

Properties of Matrix Transformations: The following theorem lists four basic properties of matrix transformations that follow from properties of matrix multiplication.

THEOREM (1) For every matrix A, the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k:

(i) $T_A(\mathbf{0}) = \mathbf{0}$

(ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ [Homogeneity Property]

(iii) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ [Additivity Property]

(iv) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

The following theorem states that if two matrix transformations from R^n to R^m have the same image at each point of R^n, then the matrices themselves must be same.

THEOREM (2) If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

Example (2) Zero Transformations

If 0 is the $m\times n$ zero matrix, then $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$

so multiplication by zero maps every vector in R^n into zero vector in R^m. We call T_0 the zero transformation from R^n to R^m.

Example (3) Identity Operators

If I is the $n\times n$ identity matrix, then $T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$

so multiplication by I maps every vector in R^n into itself. We call T_I the Identity operator on R^n.

A PROCEDURE FOR FINDING STANDARD MATRICES FOR A MATRIX TRANSFORMATION

STEP ① Find the images of standard basis vectors $e_1, e_2, \cdots, e_n$ for R^n in column form.

STEP ② Construct the matrix that has the images obtained in Step ① as its successive columns. This matrix is the standard matrix for the transformation.

REFLECTION OPERATORS : Some of the most basic matrix operators on R^2 and R^3 are those that map each point into its symmetric image about a fixed line or a fixed plane; these are called Reflection Operators. Table ① shows the standard matrices for the reflections about the co-ordinate axes in R^2 and Table ② shows the standard matrices for the reflections about the co-ordinate plane in R^3.

SEC 4.10 PROPERTIES OF MATRIX TRANSFORMATIONS

In this Section, we will discuss properties of matrix transformations. We will show, for example, that if several matrix transformations are performed in succession, then the same result can be obtained by a single matrix transformation that is chosen appropriately. We will also explore the relationship between the invertibility of a matrix and properties of the corresponding transformation.

Compositions of Matrix Transformations

Suppose that T_A is a matrix transformation from R^n to R^k and

T_B is a matrix transformation from R^k to R^m.

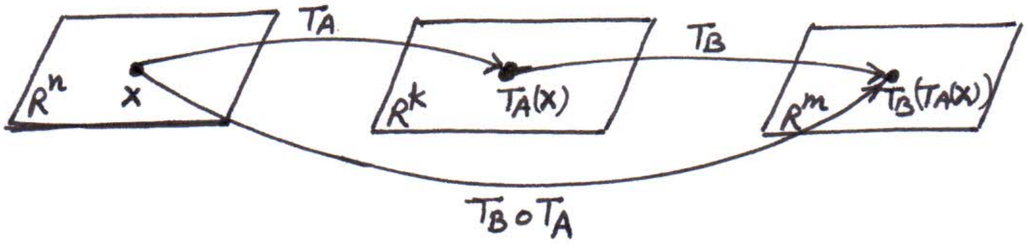
If $\mathbf{x}$ is a vector in R^n, then T_A maps this vector into a vector $T_A(\mathbf{x})$ in R^k and

T_B, in turn, maps that vector into the vector $T_B(T_A(\mathbf{x}))$ in R^m.

This process creates a transformation from R^n to R^m that we call the Composition of T_B with T_A and denote by the symbol $T_B \circ T_A$, which is read "T_B circle T_A".

As illustrated in fig., the transformation T_A in the formula is performed first; that is,

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) \qquad \text{——— (1)}$$



This composition is itself a matrix transformation since

$$\begin{aligned}(T_B \circ T_A)(\mathbf{x}) &= T_B(T_A(\mathbf{x})) \\ &= B(T_A(\mathbf{x})) \\ &= B(A\mathbf{x}) \\ &= (BA)\mathbf{x}\end{aligned}$$

which shows that it is a multiplication by BA.

This is expressed by the formula $\quad T_B \circ T_A = T_{BA} \qquad$ ——— (2)

NOTE: Just as it is not true that $AB = BA$

so it is not true, in general, that $T_B \circ T_A = T_A \circ T_B$

That is, Order matters when matrix transformations are composed.

To extend formula ② to three factors, consider the matrix transformations

$$T_A : R^n \to R^k,\quad T_B : R^k \to R^l,\quad T_C : R^l \to R^m$$

We define the composition $(T_C \circ T_B \circ T_A) : R^n \to R^m$ by

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\mathbf{x})))$$

As above, it can be shown that this is a matrix transformation whose standard matrix is CBA and that

$$T_C \circ T_B \circ T_A = T_{CBA} \qquad \text{——} \quad ③$$

We can use square brackets to denote a matrix transformation without referencing a specific matrix. Thus, for example, the formula

$$[T_2 \circ T_1] = [T_2][T_1] \qquad \text{——} \quad ④$$

is a restatement of formula ② which states that the standard matrix for a composition is the product of the standard matrices in appropriate order.

Similarly $[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$ —— ⑤

is a restatement of formula ③.

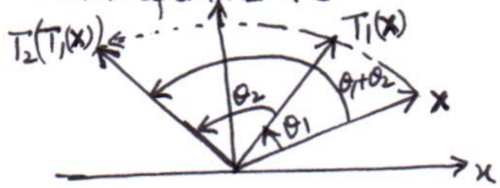
<u>Example ①</u> <u>Composition of Two Rotations</u>

Let $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ are the matrix operators that rotate vectors through the angles θ_1 and θ_2 respectively. Thus the operation

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

first rotates $\mathbf{x}$ through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2. It follows that the net effect of $T_2 \circ T_1$ is to rotate each vector in R^2 through the angle $\theta_1 + \theta_2$ (See Fig.). Thus the standard matrices for these matrix operators are

$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix},\quad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$



$$[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

These matrices should satisfy ④. We can confirm that this is so as follows —

$$[T_2][T_1] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_1\sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= [T_2 \circ T_1]$$